

The exponentiated Hencky-logarithmic strain energy.

Part III: Coupling with idealized isotropic finite strain plasticity

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Dedicated to David J. Steigmann, a great scientist and good friend

Abstract

We investigate an immediate application in finite strain multiplicative plasticity of the family of isotropic volumetric-isochoric decoupled strain energies

$$F \mapsto W_{\text{eH}}(F) := \widehat{W}_{\text{eH}}(U) := \begin{cases} \frac{\mu}{k} e^{k \|\text{dev}_n \log U\|^2} + \frac{\kappa}{2\widehat{k}} e^{\widehat{k} [\text{tr}(\log U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0, \end{cases}$$

based on the Hencky-logarithmic (true, natural) strain tensor $\log U$. Here, $\mu > 0$ is the infinitesimal shear modulus, $\kappa = \frac{2\mu+3\lambda}{3} > 0$ is the infinitesimal bulk modulus with λ the first Lamé constant, k, \widehat{k} are dimensionless fitting parameters, $F = \nabla \varphi$ is the gradient of deformation, $U = \sqrt{F^T F}$ is the right stretch tensor and $\text{dev}_n \log U = \log U - \frac{1}{n} \text{tr}(\log U) \cdot \mathbb{1}$ is the deviatoric part of the strain tensor $\log U$.

Based on the multiplicative decomposition $F = F_e F_p$, we couple these energies with some isotropic elasto-plastic flow rules $F_p \frac{d}{dt}[F_p^{-1}] \in -\partial\chi(\text{dev}_3 \Sigma_e)$ defined in the plastic distortion F_p , where $\partial\chi$ is the subdifferential of the indicator function χ of the convex elastic domain $\mathcal{E}_e(W_{\text{iso}}, \Sigma_e, \frac{1}{3}\sigma_y^2)$ in the mixed-variant Σ_e -stress space and $\Sigma_e = F_e^T D_{F_e} W_{\text{iso}}(F_e)$. While W_{eH} may loose ellipticity, we show that loss of ellipticity is effectively prevented by the coupling with plasticity, since the ellipticity domain of W_{eH} on the one hand, and the elastic domain in Σ_e -stress space on the other hand, are closely related. Thus the new formulation remains elliptic in elastic unloading at any given plastic predeformation. In addition, in this domain, the true-stress-true-strain relation remains monotone, as observed in experiments.

Key words: Hencky strain, logarithmic strain, natural strain, true strain, Hencky energy, volumetric-isochoric split, multiplicative decomposition, elasto-plasticity, bounded elastic distortions, ellipticity domain, return mapping algorithm, finite strain plasticity, isotropic formulation, 9-dimensional flow rule, associated plasticity, subdifferential formulation, convex elastic domain, plastic spin

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1 Introduction

1.1 Preliminaries

It is impossible to give an account of all works treating finite strain plasticity based in some way or another on the logarithmic strain space description. The logarithmic description¹ is arguably the simplest approach to finite plasticity, suitable for the phenomenological description of isotropic polycrystalline metals if the structure of geometrically linear theories is used with respect to the Lagrangian logarithmic strain. In this paper we do not consider hypoelastic-plastic models [72, 22, 43, 96] in which, contrary to hyperelastic models, the potential character of the elastic energy is ignored [42, 52]. Otherwise, they are simply the hyperelastic models rewritten in a suitable incremental form. In case of the logarithmic rate, however, the hypo-elastic model integrates exactly to the well-known hyperelastic quadratic Hencky model.

In isotropic finite strain computational hyperelasto-plasticity [4, 3, 19, 84, 85, 78, 13, 79] the mostly used elastic energy is the quadratic Hencky logarithmic energy [90, 57, 16, 47, 95, 97, 7, 26] (see also [24, 80, 82, 76, 74, 17, 51])

$$W_H(F_e) := \frac{\mu}{4} \|\text{dev}_n \log C_e\|^2 + \frac{\kappa}{8} [\text{tr}(\log C_e)]^2 = \mu \|\text{dev}_n \log U_e\|^2 + \frac{\kappa}{2} [\text{tr}(\log U_e)]^2, \quad (1.1)$$

where $\mu > 0$ is the infinitesimal shear modulus, $\kappa > 0$ is the infinitesimal bulk modulus, $C_e := F_e^T F_e$ is the elastic right Cauchy-Green tensor, U_e the right stretch tensor, i.e. the unique element of $\text{PSym}(n)$ for which $U_e^2 = C_e$ and

$$F = F_e \cdot F_p \quad (1.2)$$

is the multiplicative decomposition of the deformation gradient [33, 34, 35, 38, 60, 67, 9, 50, 49]. Here we have used the Frobenius tensor norm $\|X\|^2 = \langle X, X \rangle_{\mathbb{R}^{n \times n}}$, where $\langle X, Y \rangle_{\mathbb{R}^{n \times n}}$ is the standard Euclidean scalar product on $\mathbb{R}^{n \times n}$. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{1}$, so that $\text{tr}(X) = \langle X, \mathbb{1} \rangle$.

¹According to Hanin and Reiner [23, page 384]: “... there are problems in large plastic deformation. Here the only adequate measure is the Hencky measure, because this is the only measure in which the extensions form a group as can be seen from the relation

$$\log \frac{\ell_3}{\ell_1} = \log \left(\frac{\ell_3}{\ell_2} \frac{\ell_2}{\ell_1} \right) = \log \frac{\ell_3}{\ell_2} + \log \frac{\ell_2}{\ell_1}.$$

This property of forming a group is required in plasticity because in (ideal) plasticity (as in viscosity) the amount of finite deformation reached at any time is of no physical significance. As a matter of fact no definite meaning can be attached to such deformation because while in elasticity there exists an ‘unstrained state’ to which the length ℓ_0 is referred, no ‘undeformed state’ can be defined. In plasticity, as in viscosity, the increase in length $d\ell$, which takes place during the time-increment dt , can only be referred to the instantaneous length ℓ so that the extension

$$\varepsilon = \int_{\ell_n}^{\ell_{n+1}} \frac{d\ell}{\ell} = \log \frac{\ell_{n+1}}{\ell_n}$$

which is Hencky’s measure. At the same time, only in the Hencky measure can the cubical dilatation be measured by the first invariant as can be seen from

$$\varepsilon_v = \log \frac{V}{V_0} = \log \left(\frac{\ell_i}{\ell_{0i}} \frac{\ell_j}{\ell_{0j}} \frac{\ell_k}{\ell_{0k}} \right) = \log \frac{\ell_i}{\ell_{0i}} + \log \frac{\ell_j}{\ell_{0j}} + \log \frac{\ell_k}{\ell_{0k}}$$

so that the deviator is of physical significance; and plasticity relations must be expressed in terms of deviators.”

Among the works which use the Hencky strain in elasto-plasticity we may also mention [86, 47, 74, 8, 76, 75, 14, 18]. The expression W_H is the energy considered by J.C. Simo (see Eq. (3.4), page 147, from [87] and also [1]) because

$$\begin{aligned} W_H(F_e) &:= \mu \|\operatorname{dev}_n \log \sqrt{F_e^T F_e}\|^2 + \frac{\kappa}{2} [\operatorname{tr}(\log \sqrt{F_e^T F_e})]^2 = \frac{\mu}{4} \|\operatorname{dev}_n \log F_e^T F_e\|^2 + \frac{\kappa}{8} [\operatorname{tr}(\log F_e^T F_e)]^2 \\ &= \frac{\mu}{4} \|\operatorname{dev}_n \log F_e^T F_e\|^2 + \frac{\kappa}{2} [\log(\det F)]^2. \end{aligned} \quad (1.3)$$

As J.C. Simo already pointed out [88, page 392], the Hencky energy W_H “has the correct behaviour for extreme strains in the sense that” $W(F_e) \rightarrow \infty$ as $\det F_e \rightarrow 0$ and, likewise $W(F_e) \rightarrow \infty$ as $\det F_e \rightarrow \infty$, but W_H “is not a convex function of” $\det F_e$ “and hence W_H “cannot be a polyconvex function of the deformation gradient [...]. Therefore, the stored energy function” W_H “cannot be accepted as a correct model of elasticity for extreme strains. Despite this shortcoming, the model provides an excellent approximation for moderately large elastic strains, vastly superior to the usual Saint-Venant-Kirchhoff model of finite elasticity². Furthermore, this limitation has negligible practical implications in realistic models of classical plasticity, which are typically restricted to small elastic strains, and is more than offset by the simplicity of the return mapping algorithm in stress space, which takes a format identical to that of the infinitesimal theory”. The last statement is the core argument why the Hencky energy is favoured in computational metal elasticity³.

Several models of such a type have been considered in [71, 32]. The decisive advantage of using the energy W_H compared to other elastic energies stems from the fact that computational implementations of elasto-plasticity [17] based on the additive decomposition $\varepsilon = \varepsilon_e + \varepsilon_p$ in infinitesimal models [68, 15, 70, 61, 27], can be used with nearly no changes also in isotropic finite strain problems [88, page 392].

The computation of the elastic equilibrium at given plastic distortion F_p suffers, however, under the well-known non-ellipticity⁴ of W_H [5, 2, 54, 28].

Recently, it has been discovered that the elastic Hencky energy does have a fundamental differential geometric meaning, not shared by any other elastic energy. In fact

$$\begin{aligned} \widetilde{W}_H^{\text{iso}} \left(\frac{U}{\det U^{1/n}} \right) &= \operatorname{dist}_{\text{geod}}^2 \left(\frac{F}{(\det F)^{1/n}}, \operatorname{SO}(n) \right) = \operatorname{dist}_{\text{geod}, \operatorname{SL}(n)}^2 \left(\frac{F}{(\det F)^{1/n}}, \operatorname{SO}(n) \right) = \|\operatorname{dev}_n \log U\|^2, \\ \widetilde{W}_H^{\text{vol}}(\det U) &= \operatorname{dist}_{\text{geod}}^2 \left((\det F)^{1/n} \cdot \mathbb{1}, \operatorname{SO}(n) \right) = \operatorname{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbb{1}}^2 \left((\det F)^{1/n} \cdot \mathbb{1}, \mathbb{1} \right) = |\log \det F|^2, \end{aligned}$$

where $\operatorname{dist}_{\text{geod}, \mathbb{R}_+ \cdot \mathbb{1}}^2$ and $\operatorname{dist}_{\text{geod}, \operatorname{SL}(n)}^2$ are the canonical left invariant geodesic distances on the Lie-group $\operatorname{SL}(n)$ and on the group $\mathbb{R}_+ \cdot \mathbb{1}$, respectively (see [63, 62, 69, 39]). For this investigation new mathematical tools had to be discovered [69, 36] also having consequences for the classical polar decomposition [30, 29]. Here we adopt the usual abbreviations of Lie-group theory and we let $\operatorname{Sym}(n)$ and $\operatorname{PSym}(n)$ denote the symmetric and positive definite symmetric tensors respectively. We denote by $C = F^T F$ the right Cauchy-Green tensor, U the right stretch tensor, $B = F F^T$ the left Cauchy-Green (or Finger) tensor, and by V the left stretch tensor.

In the remaining part of this paper, after a paragraph giving some information on the results obtained for the exponentiated Hencky energy, we will consider the coupling to finite plasticity based on a 9-dimensional flow rule [91, 21].

²The isotropic Saint-Venant-Kirchhoff elastic energy $W_{SVK}(F_e)$ reads: $W_{SVK}(F_e) = \frac{\mu}{4} \|F_e^T F_e - \mathbb{1}\|^2 + \frac{\lambda}{8} [\operatorname{tr}(F_e^T F_e - \mathbb{1})]^2$ and does not satisfy the Baker-Ericksen-inequalities, and is not separately convex. Therefore $W_{SVK}(F_e)$ is not rank-one convex [54, 77, 83]. Moreover, in the neighbourhood of the identity it has the wrong nonlinear second order correction compared to all known experimental facts. For this reason, $W_{SVK}(F_e)$ is not a useful strain energy expression and should therefore be avoided in simulations.

³We need to be a little more specific. For the additive model in the format $\|\log C - \log C_p\|^2$ the complete systems of equations of the plastic flow rule are **identical** to the infinitesimal additive model, while for the truly multiplicative model the return mapping algorithm is **similar** to the infinitesimal case.

⁴We know that W_H is LH-elliptic in a large neighbourhood of the identity if $\lambda, \mu > 0$, $\lambda_i \in [0.21162..., 1.39561...]$ (see [5, 6]), therefore $F \mapsto W_H(F)$ is LH-elliptic for small elastic strains. Since the elasto-plastic model should secure small elastic strains anyway it seems that the non-ellipticity occurring for larger elastic strains is not essential. However, in numerical FEM-implementation it is necessary to compute the so-called elastic trial stress. The corresponding elastic trial deformation states F_e may well be far outside the LH-ellipticity range. The model using the Hencky energy W_H cannot guarantee that the computation of the elastic trial state is well-posed!

1.2 The exponentiated Hencky energy

With a view to overcome the shortcomings of the quadratic Hencky energy, in a previous work [65] we have modified the Hencky energy and we considered

$$W_{\text{eH}}(F) = W_{\text{eH}}^{\text{iso}}\left(\frac{F}{\det F^{\frac{1}{n}}}\right) + W_{\text{eH}}^{\text{vol}}(\det F^{\frac{1}{n}} \cdot \mathbb{1}) = \begin{cases} \frac{\mu}{k} e^{k \|\text{dev}_n \log U\|^2} + \frac{\kappa}{2\hat{k}} e^{\hat{k}[(\log \det U)]^2} & \text{if } \det F > 0, \\ +\infty & \text{if } \det F \leq 0. \end{cases} \quad (1.4)$$

We have called this the **exponentiated Hencky energy**. For the two-dimensional situation $n = 2$ and for $\mu > 0, \kappa > 0$, we have established that the functions $W_{\text{eH}} : \mathbb{R}^{n \times n} \rightarrow \overline{\mathbb{R}}_+$ from the family of exponentiated Hencky type energies are **rank-one convex** [65] for $k \geq \frac{1}{4}$ and $\hat{k} \geq \frac{1}{8}$, while they are **polyconvex** [66] for $k \geq \frac{1}{3}$ and $\hat{k} \geq \frac{1}{8}$.

Regarding the three-dimensional case we have proved [65] that, for all $k > 0$, the function $F \mapsto e^{k \|\text{dev}_3 \log U\|^2}$, $F \in \text{GL}^+(3)$ is not rank-one convex. However, in the next section we will discuss an interesting relation between non-ellipticity of W_{eH} in three-dimensions and finite plasticity models.

We note that the **Kirchhoff stress tensor** τ_{eH} corresponding to the exponentiated energies is given [73] by

$$\tau_{\text{eH}} = 2\mu e^{k \|\text{dev}_3 \log V\|^2} \cdot \text{dev}_3 \log V + \kappa e^{\hat{k}[\text{tr}(\log V)]^2} \text{tr}(\log V) \cdot \mathbb{1}, \quad (1.5)$$

while the **Cauchy stress tensor** is given by

$$\sigma_{\text{eH}} = e^{-\text{tr}(\log V)} \cdot \tau_{\text{eH}}.$$

Both tensors σ_{eH} and τ_{eH} differ from their classical Hencky-counterparts σ_{H} and τ_{H} only by some nonlinear scalar factors. Moreover, by orthogonal projection onto the Lie-algebra $\mathfrak{sl}(3)$ and $\mathbb{R} \cdot \mathbb{1}$, respectively, we find

$$\text{dev}_3 \sigma_{\text{eH}} = e^{-\text{tr}(\log V)} \text{dev}_3 \sigma_{\text{H}}, \quad \text{tr}(\sigma_{\text{eH}}) = e^{-\text{tr}(\log V)} \text{tr}(\sigma_{\text{H}}). \quad (1.6)$$

Therefore, the deviatoric part of the Cauchy stress $\text{dev}_3 \sigma_{\text{eH}}$ and the trace of the Cauchy stress $\text{tr}(\sigma_{\text{eH}})$ are in a simple relation with the corresponding quantities for the quadratic Hencky energy W_{H} . Hence, the change of a given FEM-implementation of W_{H} into W_{eH} is nearly free of costs [92, 40, 53, 25].

We also need to introduce the convex elastic domain in the Kirchhoff-stress space

$$\mathcal{E}_e(\tau_e, \frac{2}{3} \sigma_y^2) := \left\{ \tau_e \in \text{Sym}(3) \mid \|\text{dev}_3 \tau_e\|^2 \leq \frac{2}{3} \sigma_y^2 \right\} \subset \text{Sym}(3). \quad (1.7)$$

Incidentally, the set $\mathcal{E}_e(\tau_e, \frac{2}{3} \sigma_y^2)$ coincides with the set considered in the study of the monotonicity properties of the map $\log U \mapsto \sigma_{\text{eH}}(\log U)$ which we have called the true-stress-true-strain (TSTS-M⁺) monotonicity condition [65, 31]. The monotonicity of the Cauchy stress tensor as a function of $\log B$ or $\log V$ means

$$\langle \sigma(\log B_1) - \sigma(\log B_2), \log B_1 - \log B_2 \rangle > 0, \quad \forall B_1, B_2 \in \text{PSym}^+(3), B_1 \neq B_2, \quad (1.8)$$

which implies the **true-stress-true-strain-invertibility** (TSTS-I), i.e. the invertibility of the map $\log B \mapsto \sigma(\log B)$. This means that for our W_{eH} -formulation, the **true-stress-true-strain** relation is **monotone inside the elastic domain** $\mathcal{E}_e(\tau_e, \frac{2}{3} \sigma_y^2)$. This is a feature of W_{eH} not shared with any other known elastic energy. For more constitutive issues regarding the interesting properties of W_{eH} we refer the reader to [65].

2 Multiplicative isotropic elasto-plasticity directly in terms of the non-symmetric plastic distortion F_p

In planar elasto-plasticity⁵ our development suggests to replace the energy W_{H} by

$$\begin{aligned} W_{\text{eH}}(F_e) &:= \frac{\mu}{k} e^{\frac{k}{4} \|\text{dev}_2 \log C_e\|^2} + \frac{\kappa}{2\hat{k}} e^{\frac{\hat{k}}{4} [\text{tr}(\log C_e)]^2} \\ &= \frac{\mu}{k} e^{\frac{k}{4} \|\text{dev}_2 \log F_p^{-T} F^T F F_p^{-1}\|^2} + \frac{\kappa}{2\hat{k}} e^{\frac{\hat{k}}{4} [\text{tr}(\log F^T F)]^2} = \frac{\mu}{k} e^{\frac{k}{4} \|\text{dev}_2 \log F_p^{-T} F^T F F_p^{-1}\|^2} + \frac{\kappa}{2\hat{k}} e^{\hat{k} [\log(\det F)]^2}, \end{aligned} \quad (2.1)$$

⁵In order to model plane strain with this model, the coefficient κ has to be modified in order to be consistent with plane strain linear elasticity in the infinitesimal limit, see (2.2).

where we have imposed the condition of plastic incompressibility $\det F_p = 1$. Let us remark that for small deformations

$$\begin{aligned} W_{\text{eh}}(F_e) &= \frac{\mu}{k} e^{k \|\text{dev}_2 \log U_e\|^2} + \frac{\kappa}{2k} e^{\widehat{k} [\text{tr}(\log U_e)]^2} = \mu \|\text{dev}_2 \log U_e\|^2 + \frac{\kappa}{2} [\text{tr}(\log U_e)]^2 + \text{h.o.t.} \\ &= \mu \|\log U_e\|^2 + \frac{\kappa - \mu}{2} [\text{tr}(\log U_e)]^2 + \text{h.o.t.} = \mu \|\varepsilon_e\|^2 + \frac{\kappa - \mu}{2} [\text{tr}(\varepsilon_e)]^2 + \text{h.o.t.}, \end{aligned} \quad (2.2)$$

where ε_e is the symmetric plastic strain. A direct identification of the constitutive coefficients gives us that

$$\mu = \mu_{3D}, \quad \kappa - \mu = \lambda_{3D}. \quad (2.3)$$

Lemma 2.1. (rank-one convexity and multiplicative decomposition) *If the elastic energy $F \mapsto W(F)$ is rank-one convex, it follows that the elasto-plastic formulation*

$$F \mapsto W(F, F_p) := W(F F_p^{-1}) = W(F_e) \quad (2.4)$$

remains rank-one convex w.r.t F [57, 55, 37] at given plastic distortion F_p .

Proof. This is clear, because

$$D_F^2[W(F F_p^{-1})] \cdot (\xi \otimes \eta, \xi \otimes \eta) = D_{F_e}^2[W(F_e)] \cdot ((\xi \otimes \eta) F_p^{-1}, (\xi \otimes \eta) F_p^{-1}) = D_{F_e}^2[W(F_e)] \cdot (\xi \otimes \widehat{\eta}, \xi \otimes \widehat{\eta}),$$

where $\widehat{\eta} = F_p^{-T} \eta$. \square

Remark 2.2. *The same constitutive invariance property is true for convexity, polyconvexity and quasiconvexity [57, 32].*

Therefore, the multiplicative approach is ideally suited as far as preservation of ellipticity properties for elastic unloading is concerned. Note that this feature is not true for some additive approaches, see [64].

Definition 2.3. (reduced dissipation inequality-thermodynamic consistency) *We say that the reduced dissipation inequality along the plastic evolution is satisfied if and only if*

$$\frac{d}{dt}[W(F F_p^{-1}(t))] \leq 0, \quad (2.5)$$

for all constant in time F .

Let us further remark that for fixed F and for an energy for which the decomposition into isochoric and volumetric parts

$$W = W_{\text{iso}}(F_e) + W_{\text{vol}}(F_e) = W_{\text{iso}}(F F_p^{-1}) + W_{\text{vol}}(F) \quad (2.6)$$

holds true, in view of Sansour's result [81] (see also [46, 48, 45] and [89, page 305]), we have for the reduced dissipation inequality

$$\begin{aligned} \frac{d}{dt}[W_{\text{iso}}(F F_p^{-1})] &= \langle D_{F_e} W_{\text{iso}}(F_e), F \frac{d}{dt}[F_p^{-1}] \rangle = \langle D_{F_e} W_{\text{iso}}(F_e), F F_p^{-1} F_p \frac{d}{dt}[F_p^{-1}] \rangle \\ &= \langle F_e^T D_{F_e} W_{\text{iso}}(F_e), F_p \frac{d}{dt}[F_p^{-1}] \rangle = \langle \Sigma_e, F_p \frac{d}{dt}[F_p^{-1}] \rangle = -\langle \Sigma_e, \frac{d}{dt}[F_p] F_p^{-1} \rangle \leq 0, \end{aligned} \quad (2.7)$$

where

$$\Sigma_e = F_e^T D_{F_e} W_{\text{iso}}(F_e) = 2 C_e D_{C_e}[\widehat{W}_{\text{iso}}(C_e)] = F_e^T \tau_e F_e^{-T}$$

is the mixed variant (transformed) Kirchhoff tensor and

$$\tau_e := 2 D_{B_e}[\widehat{W}_{\text{iso}}(B_e)] B_e = 2 D_{\log B_e}[\widehat{W}_{\text{iso}}(\log B_e)] = D_{\log V_e}[W_{\text{iso}}(\log V_e)]$$

is the elastic Kirchhoff stress-tensor. Note that Σ_e is symmetric in case of elastic isotropy, while τ_e is always symmetric. The tensor $\Sigma = C \cdot S_2(C)$, where $S_2 = 2 D_C[W(C)]$ is the second Piola-Kirchhoff stress tensor, is sometimes called the **Mandel stress tensor** and $\text{dev}_3 \Sigma_e = \text{dev}_3 \Sigma_E$, where Σ_E is the elastic **Eshelby tensor**

$$\Sigma_E := F_e^T D_{F_e}[W(F_e)] - W(F_e) \cdot \mathbb{1} = D_{\log C_e}[\overline{W}(\log C_e)] - \overline{W}(\log C_e) \cdot \mathbb{1},$$

driving the plastic evolution (see e.g. [20, 60, 41, 12, 10, 11]).

A simple thermodynamically admissible perfect plasticity model [44, page 67](see also [89, 56, 58, 59, 60]) is obtained by defining the plastic evolution

$$F_p \frac{d}{dt}[F_p^{-1}] = -\frac{d}{dt}[F_p] F_p^{-1} \in -\partial\chi(\text{dev}_3 \Sigma_e), \quad (2.8)$$

where $\partial\chi$ is the subdifferential of the indicator function χ of the convex elastic domain

$$\mathcal{E}_e(\Sigma_e, \frac{1}{3}\sigma_y^2) := \{\Sigma_e \in \text{Sym}(3) \mid \|\text{dev}_3 \Sigma_e\|^2 \leq \frac{1}{3}\sigma_y^2\} \quad (2.9)$$

in the mixed-variant Σ_e -stress space⁶.

The choice (2.8) ensures $\frac{d}{dt}[W_{\text{iso}}(FF_p^{-1})] \leq 0$ at fixed F , therefore the reduced dissipation inequality (2.7) is satisfied and the deviatoric formulation together with the use of $F_p \frac{d}{dt}[F_p^{-1}] = -\frac{d}{dt}[F_p] F_p^{-1}$ as conjugate variable guarantees $\det F_p = 1$.

Next, a (for us at first surprising) algebraic estimate is introduced.

Lemma 2.4. *Let $F_e \in \text{GL}^+(3)$ be given. Then it holds $\|F_e^T S F_e^{-T}\|^2 \geq \frac{1}{2}\|S\|^2$ for all $S \in \text{Sym}(3)$, the constant being independent of F_e .*

Proof. Let us define the left Cauchy-Green tensor $B_e = F_e F_e^T \in \text{PSym}(3)$. We have

$$\|F_e^T S F_e^{-T}\|^2 = \langle F_e^T S F_e^{-T}, F_e^T S F_e^{-T} \rangle = \langle F_e F_e^T S, S F_e^{-T} F_e^{-1} \rangle = \langle B_e S, S B_e^{-1} \rangle. \quad (2.10)$$

Since $B_e = F_e F_e^T \in \text{PSym}(3)$, there is $Q \in \text{SO}(3)$ such that $D_e = Q^T B_e Q = \text{diag}(d_1, d_2, d_3)$. Hence, we obtain

$$\|F_e^T S F_e^{-T}\|^2 = \langle Q^T D_e Q S, S Q^T D_e^{-1} Q \rangle = \langle D_e Q S Q^T, Q S Q^T D_e^{-1} \rangle. \quad (2.11)$$

Moreover, considering $Q S Q^T := \hat{S} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} & \hat{S}_{13} \\ \hat{S}_{12} & \hat{S}_{22} & \hat{S}_{23} \\ \hat{S}_{13} & \hat{S}_{23} & \hat{S}_{33} \end{pmatrix}$ and using that $1 \leq \frac{d_i}{d_j} + \frac{d_j}{d_i}$ for $d_i, d_j > 0$, we deduce

$$\|F_e^T S F_e^{-T}\|^2 = \hat{S}_{11}^2 + \hat{S}_{22}^2 + \hat{S}_{33}^2 + \hat{S}_{12}^2 \left(\frac{d_1}{d_2} + \frac{d_2}{d_1} \right) + \hat{S}_{23}^2 \left(\frac{d_2}{d_3} + \frac{d_3}{d_2} \right) + \hat{S}_{13}^2 \left(\frac{d_1}{d_3} + \frac{d_3}{d_1} \right) \geq \frac{1}{2}\|\hat{S}\|^2 = \frac{1}{2}\|S\|^2,$$

due to the symmetry of \hat{S} and the proof is complete. \square

Remark 2.5. *Note that for $S \notin \text{Sym}(3)$ there is always an estimate similar to that given by Lemma 2.4 but which involves constants depending on F_e , i.e. $\|F_e^T S F_e^{-T}\|^2 \geq c(F_e) \|S\|^2$.*

Remark 2.6. *It is easy to see the relations*

$$\Sigma_e = F_e^T \tau_e F_e^{-T}. \quad (2.12)$$

Note that (2.12), as opposed to appearance, is not at variance with symmetry of Σ_e in case of isotropy.

Remark 2.7. *Since*

$$\text{dev}_3 \Sigma_e = \text{dev}_3(F_e^T \tau_e F_e^{-T}) = F_e^T \tau_e F_e^{-T} - \frac{1}{3} \text{tr}(F_e^T \tau_e F_e^{-T}) \cdot \mathbb{1} = F_e^T \left(\tau_e - \frac{1}{3} \text{tr}(\tau_e) \cdot \mathbb{1} \right) F_e^{-T}, \quad (2.13)$$

we can note

$$\text{dev}_3 \Sigma_e = F_e^T (\text{dev}_3 \tau_e) F_e^{-T}, \quad \text{dev}_3 \tau_e = F_e^{-T} (\text{dev}_3 \Sigma_e) F_e^T, \quad \text{tr}(\Sigma_e) = \text{tr}(\tau_e). \quad (2.14)$$

⁶The choice of $\frac{1}{3}$ in versus $\frac{2}{3}$ is not accidentally, see Lemma 2.4.

Thus, using Lemma 2.4 we obtain the estimate

$$\|\operatorname{dev}_3 \Sigma_e\| = \|F_e^T (\operatorname{dev}_3 \tau_e) F_e^{-T}\| \geq \frac{1}{\sqrt{2}} \|\operatorname{dev}_3 \tau_e\|, \quad (2.15)$$

which is valid for general anisotropic materials and it explains our choice of factors in $\mathcal{E}_e(\Sigma_{eH}, \frac{1}{3}\sigma_y^2)$ and $\mathcal{E}_e(\tau_{eH}, \frac{2}{3}\sigma_y^2)$, respectively. Moreover, numerical tests suggest that the LH-ellipticity domain of the distortional energy function $F \mapsto W_{eH}^{\text{iso}}(F) = \frac{\mu}{k} e^k \|\operatorname{dev}_3 \log U\|^2$, $F \in \operatorname{GL}^+(3)$, with $k \geq \frac{3}{16}$ (the necessary condition for separate convexity (SC) of $e^k \|\operatorname{dev}_3 \log U\|^2$ in 3D) is an extremely large cone

$$\mathcal{E}(W_{eH}, \text{LH}, U, 27) = \{U \in \operatorname{PSym}(3) \mid \|\operatorname{dev}_3 \log U\|^2 < 27\}; \quad (2.16)$$

Therefore we have the inclusion of domains

$$\mathcal{E}_e(\Sigma_{eH}, \frac{1}{3}\sigma_y^2) \stackrel{\text{Lemma 2.4}}{\subseteq} \mathcal{E}_e(\tau_{eH}, \frac{2}{3}\sigma_y^2) \underbrace{\subseteq}_{\text{numerical tests}} \mathcal{E}(W_{eH}, \text{LH}, U, 27).$$

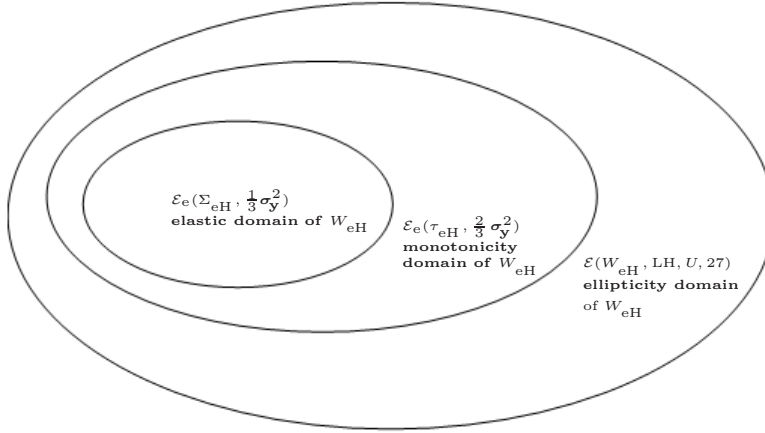


Figure 1: Elastic domains expressed in the mixed variant symmetric stress tensor Σ_{eH} and the symmetric Kirchhoff stress tensor τ_{eH} related to the ellipticity domain $\mathcal{E}(W_{eH}, \text{LH}, U, 27)$.

In (1.7) the considered convex “elastic domain”, in which monotonicity and/or ellipticity for W_{eH} is considered, is defined in terms of $\|\operatorname{dev}_3 \tau_{eH}\|$, not in terms of $\|\operatorname{dev}_3 \Sigma_{eH}\|$. However, adapting Lemma 2.4 to W_{eH} , we see that, in the three-dimensional case, our previous results indicating the loss of ellipticity only for extreme distortional strains suggest that the coupling with plasticity is most natural: permanent deformation sets in, based on a criterion of distortional energy (Huber-Hencky-von Mises-type) $\|\operatorname{dev}_3 \Sigma_{eH}\|^2 \leq \frac{1}{3}\sigma_y^2$, and our former results suggest that $W_{eH}(F_e)$ never reaches the non-elliptic domain in any elasto-plastic process. This is in sharp contrast to the loss of ellipticity of the quadratic Hencky energy W_H , which is not related to the distortional energy alone. As it turns out, for the overall non-elliptic energy W_{eH} (in three dimensions) plasticity provides a natural relaxation mechanism, which prevents loss of ellipticity in the elastic domain. Moreover, in the above defined elastic domain $\mathcal{E}_e(\Sigma_{eH}, \frac{1}{3}\sigma_y^2)$, the constitutive relation $\log B_e \mapsto \sigma(\log B_e)$ remains monotone, i.e. the true-stress-true-strain monotonicity condition (TSTS- M^+) is satisfied in $\mathcal{E}_e(\Sigma_{eH}, \frac{1}{3}\sigma_y^2)$ (see [65, 93, 94, 31]).

Remark 2.8. For the isotropic case we have $\tau_e B_e = B_e \tau_e$, which implies

$$\|\operatorname{dev}_3 \Sigma_e\|^2 = \langle F_e^T (\operatorname{dev}_3 \tau_e) F_e^{-T}, F_e^T (\operatorname{dev}_3 \tau_e) F_e^{-T} \rangle = \langle B_e (\operatorname{dev}_3 \tau_e), (\operatorname{dev}_3 \tau_e) B_e^{-1} \rangle = \|\operatorname{dev}_3 \tau_e\|^2.$$

This fact can be also proved using the fact that, in the isotropic case, both tensors Σ_e and τ_e are symmetric and they have the same invariants. Therefore, using (2.12) we obtain that in the isotropic case we have $\sqrt{2} \mathcal{E}_e(\Sigma_{eH}, \frac{1}{3}\sigma_y^2) = \mathcal{E}_e(\tau_{eH}, \frac{2}{3}\sigma_y^2)$.

Summarizing the properties of the 9-dimensional flow rule for the plastic distortion (2.8) we have:

- i) it is thermodynamically correct ($\frac{d}{dt}[W(F F_p^{-1})] \leq 0$);
- ii) the right hand side is a function of F and F_p^{-1} ;
- iii) plastic incompressibility: the constraint $\det F_p(t) = 1$ for all $t \geq 0$ follows from the flow rule, $F_p(t) \in \text{GL}^+(3)$ for all $t \geq 0$;
- iv) the above properties imply that the flow rule (2.8) is consistent;
- v) it satisfies the principle of maximum dissipation and is an associated plasticity model;
- vi) elastic unloading remains rank-one convex under arbitrary plastic predeformation.

3 Conclusion and open problems

We have shown that the multiplicative plasticity models preserve ellipticity in purely elastic processes at frozen plastic variables provided that the initial elastic response is elliptic. Preservation of LH-ellipticity is, in our view, a property which should be satisfied by any hyperelastic-plastic model since the elastically unloaded material specimen should respond reasonable under further purely elastic loading. In contrast to multiplicative models, the much used additive logarithmic model does not preserve LH-ellipticity in general [64].

An interesting question concerns the requirements that one should impose on the elastic response for arbitrary large distortional strains. One may reasonably argue that these requirements are void of any relevance, since the material can never be observed in a state of large distortional strain: prior to that, dissipative processes will occur. In the case of the energy W_{eh} , which is not rank-one elliptic for extreme distortional strains, we have explicitly shown that elastic unloading will remain rank-one convex and the true-stress-true-strain relation remains monotone. This is a remarkable feature in geometrically nonlinear material models. In a future contribution we will provide the analytical proof for the rank-one convexity domain for W_{eh} in $n = 3$.

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